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IMAGE ALGEBRA AND RANK-ORDER FILTERS

P.W. Hawkes

CEMES-LOE du CNRS, Toulouse, France

Abstract

The unifying role that is one of the attractions of image algebra is distinctly less obvious when we come to include median filters and their many close relatives (weighted median filters, rank-order filters, weighted rank-order filters). The convolutional filters, which are linear, and the morphological filters, which are not, fit in naturally and indeed have a pleasingly similar appearance in the notation of image algebra. The non-linear operation of finding the median of a set of grey-level values (or selecting any other member of a set of ranked values) is not straightforward, however, since the task of rearranging the values in ascending order is iterative, requiring a sequence of comparisons.

In this connection, a template generated by the original image to be enhanced and the window representing the zone around any given pixel, inside which the median is to be taken, proves to be very useful; this is the simplest example of the family of templates generated by suitable combination of an image and a template. The nature of such templates is the main subject of this paper.

Key Words: Rank-order filter, median filter, template, image algebra, pixel address.

* Address for correspondence:
P.W. Hawkes
CEMES-LOE du CNRS, 29 rue Jeanne Marvig,
B.P. 4347, F-31055 Toulouse Cedex, France
Telephone number: +33 562 25 78 84
FAX number: +33 562 25 79 99
E-mail: hawkes@cemes.fr

Introduction

Among the methods of enhancing an image, median filtering occupies a privileged role. Unlike the (linear) convolutional filters and the (nonlinear) morphological filters, the median filter is the simplest of an extended family of rank-order filters and their weighted counterparts, the common feature of which is that the grey-level value at a given pixel is replaced by another of the grey-level values in a window surrounding that pixel. The new value is selected by ordering the grey-level values in the window, perhaps after weighting them as a function of their distance from the pixel to be filtered, in ascending order and retaining a pre-determined member of that ordered set. In the case of median filtering, the values are not weighted and the central value is selected.

Such filters have a large literature but their representation in image algebra is less satisfactory than that of the convolutional and morphological filters. An iterative procedure is proposed in Hawkes (1995b) and it was in developing that procedure that the usefulness of the image algebra construct that is the subject of this paper emerged. The originality of this new sequence is that it creates a **template** from an image and a template whereas the usual tools of image algebra, generalized convolutions, combine an image and a template to produce a new **image**.

Image Algebra Operations

The basic object of image algebra is the **image**, which is the generic name for a host of different but familiar kinds of image: binary images, grey-level images, vector images (in which a set of values, for example, an energy-loss spectrum, is associated with each pixel), multiple-valued images (in which a single object gives rise to several images, for example, those associated with the various detectors in a scanning electron microscope) and image-valued images (in which an entire image is associated with each pixel). This last type of image is so important that is given a special name: an image-valued image is called a **template**. It is particularly useful in the representation of convolutional filters and is also important in mathematical morphology, since structuring elements can be represented by templates.

An easy way of picturing a template is provided by functions of two (vector) variables $f(\mathbf{r}, \mathbf{z})$, where $\mathbf{r} = (p, q)$ and $\mathbf{z} = (x, y)$. Then with each value of $\mathbf{z}$ (pixel address), is associated a function of $\mathbf{r}$ (an image).

The operations between a simple scalar-valued image and a template that are in routine use in image algebra all have the general character of a convolution. If **t** is a template and **a** is an image, of the form:

$$\mathbf{a} = \{(\mathbf{x},\mathbf{a}(\mathbf{x})) \mid \mathbf{x} \in \mathbf{X}\} \tag{1}$$

$$\mathbf{t} = \{(\mathbf{y},\mathbf{t}(\mathbf{y})) \mid \mathbf{y} \in \mathbf{Y}\} \tag{2}$$

$$\mathbf{t}_y \equiv \mathbf{t}(\mathbf{y}) = \{(\mathbf{x},\mathbf{t}_y(\mathbf{x})) \mid \mathbf{x} \in \mathbf{X}\} \tag{3}$$

then three generalized convolutions are defined

(i)

$$\mathbf{b} = \mathbf{b} \oplus \mathbf{t} = \{(\mathbf{y},\mathbf{b}(\mathbf{y})) \mid ...\}$$

$$\mathbf{b}(\mathbf{y}) = \sum_{x} \mathbf{a}(\mathbf{x})\mathbf{t}_y(\mathbf{x})$$

(ii)

$$\mathbf{b} = \{(\mathbf{y},\mathbf{b}(\mathbf{y})) \mid ...\}$$

$$\mathbf{b}(\mathbf{y}) = \vee\, \mathbf{a}(\mathbf{x})\mathbf{t}_y(\mathbf{x})$$

(iii)

$$\mathbf{b} = \{(\mathbf{y},\mathbf{b}(\mathbf{y})) \mid ...\}$$

$$\mathbf{b}(\mathbf{y}) = \vee\, \mathbf{a}(\mathbf{x}) + \mathbf{t}_y(\mathrm{x})$$

For our present purposes it is sufficient to assume that the pixel addresses (**x**) are the usual matrix labels of discrete arrays and that the pixel values are real numbers or integers. Other situations (and other types of image than the simple scalar-valued image assumed here) can be accommodated but we shall not consider those in this account. For extensive discussion, see Davidson (1992, 1993), Dougherty and Sinha (1995), Hawkes (1995a), Ritter (1991), and Ritter *et al.* (1990).

The first of these generalized convolutions is very similar to and in practice often identical with the everyday convolution. In such calculations, no pixel is privileged and it does not seem that any potentially useful information is lost or obscured. This is not so obviously true of the other two generalized convolutions. The max operation (or of course, min operation) selects a particular value from a set of products (ii) or sums (iii) and returns that value to the image **b**, which is the "result" of the calculation. In the language of mathematical morphology, dilation (or erosion) of an image (**a**) by a structuring element (**t**, represented by a template) generates a new image (**b**).

A simple example shows that here, however, unlike the linear convolution (i), a potentially useful piece of information **is** lost. Suppose that the template in (iii) is essentially a window so that the operation consists in selecting the largest grey-level value in a window surrounding each pixel in turn. The resulting image **b** will consist of these maximum values. What we cannot know from **b** are the addresses of these maximum values: we know what they are but not where they were! If this additional information is to be made available, the result of the calculation must not be a simple image **b**, as in (ii) or (iii) but some more elaborate type of image. If we wish to retain just the maximum value and its address then a vector-valued image might be sufficient, the grey-level values of **b** being of the form $(b_{max}, (i, j))$ where b_{max} is the maximum value and (i, j) indicates its address. Some convention would of course have to be adopted if several pixels in the window in the foregoing example reached the maximum value. It is, however, much more satisfactory in general to allow the "more elaborate type of image" to be a template so that, for each pixel of the image **a**, a (binary) image would be created indicating the site(s) of pixels reaching the maximum value; the set of such binary images would constitute a template. The equivalent in mathematical morphology would be the creation of a new structuring element from the original image and structuring element instead of just an image.

Related Work

Among the properties of median (and indeed all rank-order) filters, the fact that they commute with thresholding is of especial interest. By this we mean that if a grey-level image with a finite number of discrete grey levels (0 to 255 for example) is thresholded to form a stack of binary images and the median filter applied to each, then the result of adding the resulting (filtered) binary images is the same as that of applying the median filter directly to the original grey-level image (see, for example, Figure 72.7 in Hawkes and Kasper, 1994). This was first noticed by Nakagawa and Rosenfeld (1978) and major contributions were subsequently made by Justusson (1981), Tyan (1981), Serra (1982), and Fitch *et al*. (1984, 1985). Its interest is obvious, for the act of finding the median of a set of binary values merely requires the number of ones (or zeros) to be counted; no ordering is required. This work culminated in the analysis of Maragos and Schafer (1987), who showed formally that any median or related filter is equivalent to a maximum of morphological erosions (or a minimum of dilations) and can hence be calculated by max and min operations without sorting of grey-level values into ascending (or descending) order. A detailed examination of the relation between this way of analysing rank-order filters and that proposed in Hawkes (1995b) will be published separately. We mention the work of Maragos and Schafer (1987) here, however, not only because of its direct relevance to the problem of representing and implementing these filters efficiently but also because the interplay between structuring element (template) and the median window is directly relevant to our earlier suggestion concerning a new operation of the form image + template $\longrightarrow$ template.

Examples

The obvious example of the usefulness of an operation that generates a template from an image and a template is the median filter. As before, let **a** be a simple image and let **t** be a binary template representing a window. Then the image **b**,

$$\mathbf{b}(p,q) = \bigvee \mathbf{t}_{pq}(i,j)\mathbf{a}(i,j) \tag{4}$$

has grey-level values equal to the maximum within the window for each pixel of the image **a**. We now introduce a template **s**, which again has the form of a window but inside the window **s** has weight **b**(p,q) at the points that attain this maximum value and zero at every other point. This is easy to define:

$$\mathbf{s}_{pq}(i,j) = \mathbf{b}(p,q)\chi_{\geq_{b(p,q)}}\{\mathbf{t}_{pq}(i,j)\mathbf{a}(i,j)\} \tag{5}$$

or

$$\mathbf{s}_{pq}(i,j) = \mathbf{b}(p,q)\chi_{\geq_{b(p,q)}}(\mathbf{t}_{pq} * \mathbf{a}) \tag{6}$$

or

$$\mathbf{s} = \mathbf{b}\chi_{\geq_b}(\mathbf{t} * \mathbf{a}) \tag{7}$$

in which χ is the characteristic function.

The role of this template in the image algebra representation of median (and related) filters is explained in Hawkes (1995b). For each position of the window (template), the maximum grey-level is found. A new template is then created, the values of which are equal to the maximum if the image reaches the maximum value at that point in the window or zero otherwise. Subtraction of this from the grey-level values inside the window thus replaces the maxima by zero while leaving all the others unaffected. The process is iterated until the median (or any other member of the rank) is found. It is easy to allow for the case when more than one pixel in the window reaches the maximum value. We have chosen to illustrate the usefulness of templates generated by combining an image and a template in some suitable way with the aid of this very elementary case, since it is easily understood. Certainly, if this example in which the templates involved are simple windows were typical of the applications of such derived templates, it would be scarcely worth drawing attention to them. In practice, templates usually have a much more complex structure (structuring elements, Fourier and similar transform operators, ...) and may be adaptative and space-variant. It is with these more complicated cases in mind that we draw attention to the usefulness of derived templates.

Median filters are thus not the only examples and others will emerge as the use of image algebra becomes more commonplace. All morphological operations require finding the maximum (or minimum) of a set of values obtained in some way from those of an image and a structuring element and no record is kept of the address of the pixel at which the maximum occurred. The notion of slope transforms for morphological systems, comparable to Fourier transforms for linear convolutional cases, is still very new (Maragos 1994a,b, 1995; Dorst and van den Boomgaard, 1994a,b; van den Boomgaard and Smeulders, 1994) but it can be anticipated that the type of template examined here will be needed there. These transforms arise when we enquire whether the fact that the Fourier transform maps convolution products into direct products has any analogue for "generalized" convolutions of the type represented by (iii) above. It is found that there is indeed such a transform but the image algebraic representation has not yet been analysed in detail. It can however, be safely anticipated that the transform can be represented in terms of a template as in the better known case of the Fourier transform.

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Discussion with Reviewers

W.O. Saxton: Can you say whether existing image algebra packages allow new templates such as you envisage to be created and used?
Author: Certainly the tools needed are present in these packages. A little work will be required to create each new template.

W.O. Saxton: Do any of the interesting results about the commuting of ranking and multiple thresholding continue to hold for the ranking operators involving weighting according to distance from the pixel under consideration?
Author: Weighting is accomplished by repeating the weighted grey-level as many times as the value of the weight. The commuting rule therefore does not survive in its original form.

N. Bonnet: Several standard implementations of the median filter are used in the image processing software: The ranking method (in which no use is made of the results obtained for the previous position of the moving window) and the histogram-based method, which allows reusability and is hence much faster, especially when the neighbourhood considered is large. Is your image algebra implementation related more specifically to one of these two algorithms or is it completely independent? How does it compare in terms of computational complexity (i.e., in terms of computational load)?
Author: It can be used equally easily for either algorithm. I think that the computational load is much the same!

N. Bonnet: Median-like nonlinear filters are now beginning to be used for multi-valued images (multispectral images, for instance). Do you think that your approach can be generalized to handle such situations?
Author: Certainly, in principle at least. But it would be necessary to consider specific situations to give a more helpful answer.

N. Bonnet: Please give a list of the available software packages for image processing using image algebra.
Author: See the new handbook (Ritter and Wilson, 1996) or the Proceedings of the earlier SPIE Proceedings volumes on this subject: volumes 1350, 1568, 1769, 2030, 2300 and 2568 (1990–1995).

Additional Reference

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